HILBERT SPACE METHODS IN THE THEORY OF LIE ALGEBRAS(1)

BY

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Introduction. In the study of the complex finite-dimensional semi-simple Lie algebras a crucial role is played by the fundamental bilinear form $\langle x, y\rangle = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y))$. Since the definition is meaningless when the restriction of finite-dimensionality is removed, if any of the highly desirable properties of the form are to be retained in this case they must necessarily be given *a priori*. By reconsidering the finite-dimensional situation it is possible to formulate suitable conditions in a more convenient form. To see this let L be a complex finite-dimensional semi-simple Lie algebra and let L_0 be a compact real form for L with σ as the associated involution (conjugation). If we let $x^* = -\sigma(x)$ and $(x, y) = \langle x, y^*\rangle$ then L becomes a finite-dimensional Hilbert space, the mapping x into x^* is a Hilbert space conjugation, and the connecting property $([x, y], z) = (y, [x^*, z])$ holds for all x, y, z. An L^* algebra as defined here is simply a Lie algebra whose vector space is a Hilbert space such that the connecting property above holds. This paper is a study of such algebras with emphasis, of course, on the infinite-dimensional ones. For finite dimensions nothing new is obtained and it is shown here that in this case every semi-simple L^* algebra arises essentially from a construction like that above (see the remark after 2.5).

There is an associative algebra analogue of this problem in the paper of Ambrose [1] on H^* algebras and some of his results are used here. Any H^* algebra gives rise to an L^* algebra by letting $[x, y] = xy - yx$ and the only known examples of L^* algebras are those obtained as Lie subalgebras of H^* algebras.

The main result of this paper is a classification of the (separable) simple L^* algebras which have Cartan decompositions (see §2) and it is shown that this class coincides with the simple self-adjoint Lie subalgebras of a (separable) simple H^* algebra. The results turn out to be the natural extensions of the finite-dimensional theory.

Associated with each of the Lie algebras considered here there is a gener-

Received by the editors April 9, 1959.

(1) The results in this paper coincide in part with the author's doctoral thesis written at Massachusetts Institute of Technology under the guidance of Professor Kenkichi Iwasawa. The author would like to acknowledge his gratitude to Professor Iwasawa for his considerable patience and interest and his frequent helpful suggestions.

alized analytic group nucleus. For a discussion of this relationship one may refer to the paper of Birkhoff [2].

1. **Preliminaries.**

DEFINITION. An L^* algebra is defined as a Lie algebra L over the complex field such that the vector space of L is a Hilbert space and for each $x \in L$ there is an x^* in L with $([x, y], z) = (y, [x^*, z])$ for all y, z in L.

EXAMPLES. Let A be an H^*algebra and let $[x, y] = xy - yx$. Any closed Lie subalgebra of A which is closed under the operation of taking adjoints is then an L^*algebra. Any complex finite-dimensional semi-simple Lie algebra is an L^* algebra. The Hilbert space direct sum of L^* algebras defines an L^* algebra in the obvious way.

DEFINITIONS AND REMARKS. L will represent an L^* algebra. For subsets M, N of L let $M^* = \{m^*: m \in M\}$, $M^\perp = \{x: (x, m) = 0$ for all m in $M\}$, $[M, N]$ = the closed subspace spanned by $\{[m, n]: m \in M, n \in N\}$. For subspaces S_1 and S_2 of L the notation $S_1 + S_2$ will be used only when $S_1 \perp S_2$.

For x in L let D_x denote the linear operator $D_x y = [x, y]$. Then D_x, D_x^* are everywhere defined (this implies both are bounded) and $D_x^* = D_{x^*}$. By using the principle of uniform boundedness it is not hard to show that the mapping x to D_x is continuous from L into the space of bounded operators on L under the uniform norm. Furthermore we may assume that $\|D_x\| \leqq \|x\|$.

L will be called semi-simple if and only if $L = [L, L]$ and this is equivalent to the mapping x to D_x being one-one. L will be called simple if and only if there are no nontrivial closed ideals. It is a simple argument to show that a closed subspace I of L is an ideal of L if and only if $I^\perp$ is an ideal. Using this one obtains the result that every L^* algebra is the direct sum of an abelian ideal (the center) and a semi-simple ideal (the derived algebra, $[L, L]$). Hence an L^* algebra is necessarily *reductive* in the sense of [3, Exposé 7].

From now on we will assume L is semi-simple. Using the fact that the adjoint representation is then faithful and the properties of adjoints for operators it follows that the mapping x to x^* is involutory, conjugate linear, and $[x, y]^* = [y^*, x^*]$. Then the connecting property implies $(x, [y, z]) = ([y, z]^*, x^*)$ for all x, y, z. By semi-simplicity, $(x, y) = (y^*, x^*)$ for all x, y so that the $*$ mapping is a Hilbert space conjugation. L is then the complexification of the real Lie algebra formed by the skew-adjoint elements. It can be proved from all of this that every closed ideal of L is an L^* algebra.

A Cartan subalgebra of a semi-simple L is defined as a maximal self-adjoint abelian subalgebra. An application of Zorn's Lemma shows that every $x \in L$ with $[x, x^*] = 0$ is contained in a Cartan subalgebra. A Cartan subalgebra is necessarily closed.

1.1. Let H be a Cartan subalgebra of L. Then H is maximal abelian and $H^\perp = [H, L]$.

Proof. Suppose $[H, x] = 0$. Then $[H, x^*] = [H^*, x]^* = [H, x]^* = 0$. Hence $[H, x + x^*] = [H, x - x^*] = 0$. Since H is maximal self-adjoint abelian this implies $x+x^*$ and $x-x^*$ are in H, hence $x\in H$ and H is maximal abelian. If $h_1, h_2\in H$ and $x\in L$ then $(h_1, [h_2, x])=([h_2^*, h_1], x)$ implies $[h_2, x]\in H^\perp$ and $[H, L]\subset H^\perp$. If $x\in [H, L]^\perp$ then $(x, [h^*, y])=0$ for all y implies $([h, x], y)=0$ so that $[H, x]=0$ and $x\in H$.

In the event that L is finite-dimensional a Cartan subalgebra H as defined here is a Cartan subalgebra in the usual sense. For H is maximal abelian and for each $h\in H$, $[h, h^*]=0$ implies D_h is normal, hence diagonalizable. These two properties characterize the Cartan subalgebras of L (see [3, Exposé 9]). Conversely, if L is semi-simple and finite-dimensional, a Cartan subalgebra H of L in the sense of [3] is one in our sense for a suitable * mapping and inner product, for by Exposé 11 of [3] there is a compact real form L_0 of L with associated involution σ such that $\sigma(H)=H$. Applying the construction used in the introduction gives the result.

1.2. THEOREM 1. *Let L be a semi-simple L^* algebra. Then there exist simple closed L^* ideals L_j, indexed by some set J, such that $L=\sum_{j\in J} L_j$, the sum being the usual Hilbert space direct sum. Every closed ideal of L is obtained by summing the L_j over some subset of J.*

Outline of Proof. Let H be a Cartan subalgebra of L and B the C^* algebra generated by $\{D_h: h\in H\}$. B is then topologically and algebraically isomorphic with the algebra of all continuous complex-valued functions vanishing at infinity on the locally compact space Δ of all homomorphisms of B onto the complex numbers. Each $\alpha\in\Delta$ defines a bounded linear functional on H and hence there is a unique $h_\alpha\neq 0$ in H such that $\alpha(D_h)=(h, h_\alpha)$ for all h. Then $\|h_\alpha\|\leqq 1$ and $\alpha(D_h^*)=[\alpha(D_h)]^-$ implies $h_\alpha^*=h_\alpha$. For $\alpha, \beta\in\Delta$ let $(\alpha, \beta)=(h_\alpha, h_\beta)$ and definite $\alpha\perp\beta$ if and only if $(\alpha, \beta)=0$. A subset M of Δ will be called indecomposable if M cannot be written as the union of nonempty orthogonal subsets. Then each $\alpha\in\Delta$ is contained in a unique maximal indecomposable subset M_α. Then either $M_\alpha=M_\beta$ or $M_\alpha\perp M_\beta$. Let $\{M_j: j\in J\}$ be the set of the distinct M_α's. For each j let H_j be the span of the h_α where α runs over M_j and let $L_j=H_j+[H_j, L]$. By a proof like that used in the finite-dimensional case each L_j is a simple closed ideal of L and $L_j\perp L_k$ for $j\neq k$. If $K=\sum L_j$ then K is a closed ideal containing H (the h_α's span H) and hence $[K^\perp, H]=0$ implies $K^\perp=0$ so that $L=\sum L_j$. The last statement is a consequence of the way the decomposition is obtained.

2. **Roots and Cartan decompositions.**

DEFINITION. For this section L is a semi-simple L^* algebra with H as a Cartan subalgebra. For a linear mapping α of H into the complex numbers let $V_\alpha=\{v: [h, v]=\alpha(h)v \text{ for all } h\in H\}$. Then V_α is a closed subspace of L and α will be called a root (relative to H) if and only if $V_\alpha\neq 0$. The zero function is a root and $V_0=H$. If α is a root then necessarily it corresponds to a homomorphism of the operator algebra generated by $\{D_h: h\in H\}$. Hence α is bounded and $\alpha(h^*)=[\alpha(h)]^-$. As in the proof of Theorem 1 there is a unique

h_α in H with $\|h_\alpha\| \leqq 1$, $h_\alpha^* = h_\alpha$, and $\alpha(h) = (h, h_\alpha)$ for all h. From this it follows that if α is a root $-\alpha$ is also one and $V_\alpha^* = V_{-\alpha}$. If α, β are distinct then $V_\alpha \perp V_\beta$. By the Jacobi identity $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$.

Let $K = \sum V_\alpha$, the sum being taken over the distinct roots relative to H. Then K is a closed L^* subalgebra of L with $H \subset K \subset L$. We will say that L has a Cartan decomposition (relative to H) if and only if $K = L$, i.e. if and only if the set $\{D_h : h \in H\}$ is simultaneously diagonalizable. It is an open question as to whether or not every L has such a decomposition; however, I hope to have more complete results to be given in a later paper. Theorem 2 below settles the question if L is embedded in an H^* algebra and the later classification theory shows this is necessary as well as sufficient, at least when every simple ideal component of L is separable.

2.1. Let L be a simple L^* algebra and suppose ϕ is a continuous linear mapping of L into a Hilbert space K with $(\phi([x, y]), \phi(z)) = (\phi(y), \phi([x^*, z]))$ for all x, y, z in L. Then there is an $\epsilon \geqq 0$ such that $(\phi(x), \phi(y)) = \epsilon(x, y)$ for x and y in L.

Proof. Since ϕ is bounded there is a bounded operator B on L such that $(\phi(x), \phi(y)) = (Bx, y)$. Then $B \geqq 0$ implies B is self-adjoint. The assumption on ϕ implies B commutes with every D_x; by the spectral theorem every projection in the spectral resolution of B commutes with every D_x. The range of such a projection is then a closed ideal of L, hence is either 0 or all of L so that $B = \epsilon 1$ for some $\epsilon \geqq 0$.

2.2. THEOREM 2. *Suppose L is a semi-simple L^* subalgebra of an H^* algebra A and H is a Cartan subalgebra of L. Then L has a Cartan decomposition relative to H.*

Proof. Let $L = \sum L_j$ where each L_j is a simple closed ideal. If $H_j = H \cap L_j$ it is easily seen that H_j is a Cartan subalgebra of L_j. Hence it will be sufficient to prove the theorem when L is simple.

If I is a simple (associative) ideal of A the restriction to L of the projection P of A onto I satisfies the hypotheses of 2.1 and hence there is an $\epsilon \geqq 0$ such that $(Px, Py) = \epsilon(x, y)$ for all x, y in L. Since A is a direct sum of such simple ideals there must be some I such that the corresponding ϵ is positive. Thus L is topologically and algebraically isomorphic with a Lie subalgebra of I so that we may assume A itself is simple. Then by [1], A is the set of all Hilbert-Schmidt operators on some Hilbert space $\mathfrak{H}$.

The set H is then a collection of commutative completely continuous normal operators on $\mathfrak{H}$ and hence can be simultaneously diagonalized. Using a basis of $\mathfrak{H}$ composed of common eigenvectors for H and regarding A as the algebra of square-convergent matrices relative to this basis, H becomes a subset of the diagonal matrices. For $h \in H$ and $y \in A$ let $T_h y = hy - yh$. Then, as in the finite-dimensional case, the operators T_h can be simultaneously diagonalized. Since L is an invariant subspace under the set of all T_h and the restriction of T_h to L is D_h then L has a Cartan decomposition relative to H.

For the remainder of this section we will assume only that L is semi-simple and H is a Cartan subalgebra.

2.3. If α is a nonzero root V_α is one-dimensional.

Proof. Choose $v_1 \in V_\alpha$ with $\|v_1\|=1$. Let $v_2 \in V_\alpha$ with $(v_1, v_2)=0$. It is sufficient to show that this implies $v_2=0$. For any $v\in V_\alpha$ we have $[v_1, v^*]\in H$. For any $h\in H$, $(h, [v_1, v^*])=([h, v], v_1)=(h, h_\alpha)(v, v_1)$ implies $[v_1, v^*]=(v_1, v)h_\alpha$ so that $[v_1, v_2^*]=0$. The same argument can be used to show that $[v_2, v_2^*]=\|v_2\|^2 h_\alpha$. Then, by the Jacobi identity and the connecting property, $0=([v_1^*, v_2], [v_1^*, v_2])=([v_2, v_2^*], [v_1, v_1^*])+([v_1, v_2], [v_1, v_2])=\|v_2\|^2\|h_\alpha\|^2 +\|[v_1, v_2]\|^2$ so that $\|v_2\|=0$ and $v_2=0$.

Definition. Let R be the set of nonzero roots relative to H. By Zorn's lemma it is possible to decompose R as $R=R_1\cup R_2$ where R_1, R_2 are disjoint and $\alpha\in R_1$ if and only if $-\alpha\in R_2$. For each $\alpha\in R_1$ choose $e_\alpha\in V_\alpha$ such that $\|e_\alpha\|=1$. Then $e_\alpha^*\in V_{-\alpha}$ and $\|e_\alpha^*\|=1$. For $\alpha\in R_2$ let $e_\alpha=e_{-\alpha}^*$. Thus $e_\alpha^*=e_{-\alpha}$ for all α in R and the set $\{e_\alpha\}$ is an orthonormal set. By the proof of 2.2, $[e_\alpha, e_\alpha^*]=h_\alpha$.

Suppose $\alpha, \beta\in R$ and $\beta\neq-\alpha$. If $\alpha+\beta$ is a root let $c_{\alpha,\beta}$ be defined by the equation $[e_\alpha, e_\beta]=c_{\alpha,\beta}e_{\alpha+\beta}$, otherwise let $c_{\alpha,\beta}=0$ and $e_{\alpha+\beta}=0$.

If β is any root and α a nonzero root the sequence $\{\beta-k\alpha: k=0, \pm 1, \cdots\}$ contains only finitely many roots for if $\beta-k\alpha$ is a root then $1\geqq\|h_{\beta-k\alpha}\| =\|h_\beta-kh_\alpha\|\geqq|k|\,\|h_\alpha\|-\|h_\beta\|$. Thus it is possible to define the integers $k_1(\alpha,\beta)$ and $k_2(\alpha,\beta)$ by the conditions $\beta+k\alpha$ is a root for $-k_1\leqq k\leqq k_2$ while $\beta-(k_1+1)\alpha$ and $\beta+(k_2+1)$ are not roots. Then, by the same proof as used in [3], $(h_\alpha, h_\beta)=(1/2)[k_1(\alpha,\beta)-k_2(\alpha,\beta)]\|h_\alpha\|^2$ for any roots α, β with $\alpha\neq 0$.

2.4. Suppose $\alpha_1, \cdots, \alpha_k\in R$. Let M be the set of all roots which are linear combinations with integral coefficients of $\alpha_1, \cdots, \alpha_k$. Let V be the span of the e_α's where $\alpha\in M$ and let H_1 be the span of $h_{\alpha_1}, \cdots, h_{\alpha_k}$. Then $L_1=H_1+V$ is a finite-dimensional semi-simple L^* algebra with H_1 as a Cartan subalgebra and M is the complete set of roots relative to H_1.

Proof. The proof is straightforward except, perhaps, for the statement that the dimension of L_1 is finite. Since $\dim H_1\leqq k<\infty$, $\dim L_1$ is infinite if and only if $\{e_\alpha: \alpha\in M\}$ is infinite and this can occur only if $\{h_\alpha: \alpha\in M\}$ is infinite. In this event the latter set is an infinite bounded set in the unitary space H_1 and must then contain an infinite convergent sequence h_{β_n}. Letting $h_i =2\|h_{\alpha_i}\|^{-2}h_{\alpha_i}$ for $i=1, \cdots, k$, (h_{β_n}, h_i) is an integer for all n and i and H_1 is spanned by $h_1, \cdots, h_k$. From this it is clear that no such sequence exists and consequently L_1 is finite-dimensional.

2.5. Suppose L is finite-dimensional and simple. Let $\langle x, y\rangle=\mathrm{Tr}(D_xD_y)$ for all x, y. Then there is an $\epsilon>0$ such that $\langle x, y\rangle=(x, y^*)$.

Proof. Define the operator B on L by the equation $(Bx, y)=\langle x, y^*\rangle$ for all

x, y. Then $(Bx, x) = \mathrm{Tr}(D_x D_x^*)$ implies B is positive definite. The condition $\langle [x, y], z\rangle = \langle x, [y, z]\rangle$ implies B commutes with every D_x and, by the argument used in 2.1, B must be a positive multiple of the identity.

REMARK. The result of 2.5 justifies the remarks of the introduction for finite-dimensional L^* algebras. If L is simple and ϵ is as in 2.5 let L_0 be the set of skew-adjoint elements of L and $\sigma(x) = -x^*$ for all x. Then σ is an involution and 2.5 shows that L_0 is a compact real form for L. The extension to semi-simple algebras is immediate. An immediate consequence of this relationship is the result 2.6 below which will be needed in the later classification theory.

2.6. Let L be finite-dimensional and simple with H as a Cartan subalgebra.

(i) If $\alpha, \beta, \gamma \in R$ and $\alpha+\beta+\gamma=0$ then $c_{\alpha,\beta} = c_{\beta,\gamma} = c_{\gamma,\alpha}$.

(ii) If $\alpha, \beta, \gamma, \delta \in R$, $\alpha+\beta+\gamma+\delta=0$, and the sum of no pair is zero, then $c_{\alpha,\beta}c_{\gamma,\delta} + c_{\beta,\gamma}c_{\alpha,\delta} + c_{\gamma,\alpha}c_{\beta,\delta} = 0$.

(iii) If $\alpha, \beta \in R$ and $\beta \neq -\alpha$ then

$$c_{\alpha,\beta}c_{-\alpha,-\beta} = -(1/2)k_2(\alpha, \beta)(1 + k_1(\alpha, \beta))\|h_\alpha\|^2.$$

Proof. See [3, Exposé 11, Lemmas 1, 2, 3].

3. **The classification theory.** For this section L will be a simple infinite-dimensional L^* algebra and H a Cartan subalgebra such that L has a Cartan decomposition relative to H. We further require that the space of L be separable.

DEFINITION. For a finite subset $F = \{\alpha_1, \cdots, \alpha_k\}$ of R let $L(F)$ denote the finite-dimensional semi-simple algebra defined in 2.4. Then $F_1 \subset F_2$ implies $L(F_1) \subset L(F_2)$. A subset G of R will be called a root system if and only if $\alpha \in G$ implies $-\alpha \in G$ and $\alpha, \beta \in G$, $\alpha+\beta \in R$ implies $\alpha+\beta \in G$. Then $L(F)$ is the subalgebra generated by the e_α where α ranges over the root system generated by F. Using the notion of indecomposability as in the proof of Theorem 1, if F is an indecomposable finite subset of R then the root system generated by F is indecomposable and $L(F)$ is simple. Furthermore it is clear that R is indecomposable since L is simple. A subset $\alpha_0, \cdots, \alpha_n$ of R will be called a chain from α_0 to α_α if $(h_{\alpha_{i-1}}, h_{\alpha_i}) \neq 0$ for $i = 1, \cdots, n$. Since R is indecomposable any $\alpha, \beta \in R$ must be connected by a finite chain. Any chain is indecomposable.

3.1. For any finite subset F of R there exists a finite indecomposable root system containing F.

Proof. Let $F = \{\alpha_1, \cdots, \alpha_n\}$. For each i, $1 \leqq i \leqq n-1$, let F_i be a chain from α_i to α_{i+1}. Let $F_1 = \bigcup F_i$. Then F_1 is indecomposable and finite. If F_2 is the root system generated by F_1, F_2 is indecomposable and 2.4 implies F_2 is finite.

DEFINITION. Since L is separable the orthonormal set $\{e_\alpha : \alpha \in R\}$ is countable and hence R is countably infinite. Let $R=\{\alpha_1, \alpha_2, \cdots\}$ and let $F_n = \{\alpha_1, \cdots, \alpha_n\}$.

3.2. There is a sequence G_n of finite subsets of R such that the following are true:

(i) $F_n \subset G_n \subset G_{n+1}$.

(ii) G_n is an indecomposable root system.

(iii) $R = \bigcup G_n$.

(iv) The simple subalgebras $L(G_n)$ form a strictly increasing sequence with $L=$ closure of $\bigcup L(G_n)$. All of the $L(G_n)$ are of the same Cartan type A, B, C, or D.

Proof. The sequence $\{G_n\}$ can be defined inductively. Let G_1 be a finite indecomposable root system containing F_1. Having chosen $G_1, \cdots, G_{n-1}$ satisfying (i) and (ii) let $F = G_{n-1} \cup F_n$ and choose G_n to be a finite indecomposable root system containing F. The G_n obtained in this way will then satisfy (i) and (ii). Since $R = \bigcup F_n$, (iii) will hold and $G_n \subset G_{n+1}$ implies $L(G_n) \subset L(G_{n+1})$. An $h \in H$ such that $(h, h_\alpha)=0$ for all α in R would then have $D_h=0$, hence $h=0$ and H is spanned by the set of h_α. Since the set of e_α spans $H^\perp$ then $L=$ closure of $\bigcup L(G_n)$. Now dim L is infinite and each $L(G_n)$ is finite-dimensional so that there are infinitely many distinct $L(G_n)$. Then any infinite subsequence of the G_n will also satisfy (i)–(iii) and the first part of (iv). By passing to subsequences if necessary it is possible to eliminate any duplications and furthermore obtain a sequence whose elements are all of the same type. Since their dimensions are unbounded there can be no exceptional algebras.

Definition. Let K_n be the real linear subspace of the conjugate space of H spanned by $\{\alpha: \alpha \in G_n\}$. Let $p_1 = \dim K_1$ and $p_n = \dim(K_n/K_{n-1})$ for $n=2, 3, \cdots$. Then each p_i is a positive integer and the rank of the simple algebra $L(G_n)$ is $p_1 + \cdots + p_n$.

Since G_1 is a root system for $L(G_1)$ there exist $\alpha_{1,1}, \cdots, \alpha_{1,p_1}$ in G_1 which form a linear basis of K_1. Since G_2 is a root system for $L(G_2)$ there exist $\alpha_{2,1}, \cdots, \alpha_{2,p_2}$ in G_2 such that $\alpha_{1,1}, \cdots, \alpha_{1,p_1}, \alpha_{2,1}, \cdots, \alpha_{2,p_2}$ form a linear basis for K_2. Necessarily $\alpha_{2,i} \notin G_1$. Continuing this process we can find, for each $n \geqq 2$, $\alpha_{n,1}, \cdots, \alpha_{n,p_n} \in G_n - G_{n-1}$ such that the set

$$\{\alpha_{i,j}: i = 1, \cdots, n; j = 1, \cdots, p_i\}$$

is a linear basis for K_n. Order this basis as follows:

$$\alpha_{n,p_n}, \cdots, \alpha_{n,1}, \alpha_{n-1,p_{n-1}}, \cdots, \alpha_{n-1,1}, \cdots, \alpha_{2,1}, \alpha_{1,p_1}, \cdots, \alpha_{1,1}.$$

Suppose $\tau \in K_n$, $\tau \neq 0$. Then τ is a linear combination with real coefficients of the elements of this ordered basis. Define $\tau > 0$ or $\tau < 0$ according as the first nonzero coefficient is positive or negative. For $\tau_1 \neq \tau_2$ let $\tau_1 > \tau_2$ if and only if $\tau_1 - \tau_2 > 0$. This then gives a total ordering of K_n and induces an ordering of G_n. By the choice of basis for each K_n, for integers n, m and $\alpha, \beta \in G_n \cap G_m$, $\alpha > \beta$ in the ordering of G_n if and only if $\alpha > \beta$ in the ordering of G_m.

Now suppose α, β are any roots. Choose n such that $\alpha, \beta \in G_n$ and define $\alpha > \beta$ if and only if they are so related in the ordering of G_n. This gives a well-defined total ordering on the set of all roots and has the following properties:

(i) $\alpha > 0$ implies $-\alpha < 0$.

(ii) $\alpha > 0$, $\beta > 0$ implies $\alpha + \beta > 0$.

(iii) If $\alpha > 0$ and $\alpha \notin G_n$ then $\alpha > \beta$ for every $\beta \in G_n$.

(iv) The ordering induced on G_n is a lexicographical ordering with respect to a basis of roots.

Let R^+ be the set of positive roots. Then, since G_n is finite, property (iii) implies that R^+ is well-ordered. An $\alpha \in R^+$ will be called simple if α cannot be written as the sum of two positive roots. Let S denote the set of all simple roots.

3.3. (1) $S \cap G_n$ is a complete set of simple roots (in the sense of [3]) for $L(G_n)$.

(2) For α, β in S, $\alpha - \beta$ is a root only if $\alpha = \beta$. Thus $k_1(\alpha, \beta) = k_1(\beta, \alpha) = 0$.

(3) S is linearly independent over the reals and every α in R^+ is a linear combination of elements of S with non-negative integral coefficients which are almost all zero.

(4) If $\tau = \sum n_i \alpha_i$ where $\alpha_i \in S$ and almost all n_i are zero there is an algorithm to determine whether or not τ is a root. To apply the algorithm it is sufficient to know (h_α, h_β) for all $\alpha, \beta \in S$.

Proof. (1) If $\alpha, \beta, \gamma \in R^+$ and $\alpha = \beta + \gamma$ then $\alpha > \beta > 0$ and $\alpha > \gamma > 0$. If $\alpha \in G_n$ then (iii) of the definition above implies $\beta, \gamma \in G_n$. Hence an $\alpha \in G_n$ is simple in G_n if and only if α is simple in R.

(2) and (3) can be deduced from the corresponding properties for the finite-dimensional case proved in [3, Exposé 10].

(4) Choose n such that $\tau \in K_n$. The statement then follows from the result proved in [3, Exposé 16], applied to the algebra $L(G_n)$, using the fact that the fundamental bilinear form is determined up to a constant multiple.

Definition. Define the graph of S to be the set G of all (h_α, h_β) where α, β vary over S. Then knowing the graph is equivalent to determining $\|h_\alpha\|$ and $k_2(\alpha, \beta)$ for α, β in S. If L, L' are two algebras of the type considered in this section with H and H' as Cartan subalgebras and G, G' as the corresponding graphs we will say that G is isomorphic to G' if and only if there is a mapping α to α' of S onto S' with $(h_\alpha, h_\beta) = (h'_{\alpha'}, h'_{\beta'})$ for all α, β in S.

3.4. Let L, L' be as above and suppose G is isomorphic to G'. Then there is an algebraic isomorphism ϕ of L onto L' such that:

(1) $\phi(h_\alpha) = h'_{\alpha'}$ for all $\alpha \in R$.

(2) $\phi(x)^* = \phi(x^*)$ for all $x \in L$.

(3) $(\phi(x), \phi(y)) = (x, y)$ for all x, y in L.

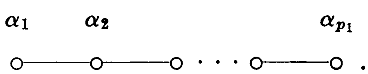
Proof. By using the algorithm of 3.3, (4) it is possible to extend the map of S onto S' to a mapping α to α' of R onto R' which preserves inner products for the h_α. This mapping then necessarily preserves all of the algebraic structure of R. For a complex linear combination $h=\sum c_i h_{\alpha_i}$, where $\alpha_i \in R$, let $\phi(h)=\sum c_i h'_{\alpha_i'}$. Then ϕ is well-defined and preserves inner products so that it extends uniquely to an isometry of H onto H' and satisfies (1). Since $\phi(h_\alpha^*)=\phi(h_\alpha)^*$ for all h_α, ϕ will satisfy (2) for any $x\in H$.

Let $\{f_{\alpha'}: \alpha' \in R'\}$ be a fixed set of elements in L' with $f_{\alpha'} \in V_{\alpha'}$, $\|f_{\alpha'}\|=1$, and $f_{\alpha'}^* = f_{-\alpha'}$. Let $c_{\alpha',\beta'}$ be the structure constants for L' defined by the set of $f_{\alpha'}$. To extend ϕ to all of L with the required properties it is then sufficient to find a set $\{e_\alpha: \alpha\in R\}$ in L with $e_\alpha \in V_\alpha$, $\|e_\alpha\|=1$, $e_\alpha^* = e_{-\alpha}$, and such that the structure constants $c_{\alpha,\beta}$ for L defined by this set satisfy $c_{\alpha,\beta}=c_{\alpha',\beta'}$.

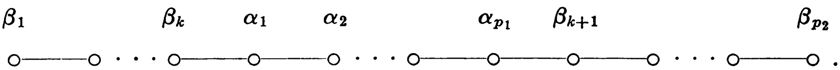
Thus the problem is reduced to finding the set of e_α. A corresponding result appears in [3, Exposé 11, Théorème 1]. An examination of the proof there shows the essential features are a well-ordering of R^+ compatible with the algebraic structure and the relations on the structure constants which were proved here in 2.6. (These hold for L since there is always an n such that α, β, γ, δ all lie in G_n.) Using these, the proof in [3] can be repeated here word for word.

3.5. Because of 3.4 it only remains to determine the possible graphs for L and give examples of each type in order to complete the classification.

First, suppose all of the $L(G_n)$ in 3.2 are of type A. Then the root diagram for the simple system $S\cap G_1$ has the form:

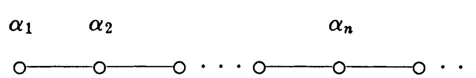


This means, of course, that $k_2(\alpha_i, \alpha_{i-1})=k_2(\alpha_i, \alpha_{i+1})=1$ and otherwise $k_2(\alpha_i, \alpha_j)=0$. Furthermore, by the remark after 2.5, $\|h_{\alpha_i}\|=\|h_{\alpha_j}\|$ for $1\leqq i$, $j\leqq p_1$. Now let $S\cap G_2$ be written as $\alpha_1, \cdots, \alpha_{p_1}, \beta_1, \cdots, \beta_{p_2}$. After any necessary reordering the diagram for $S\cap G_2$ will have the form:



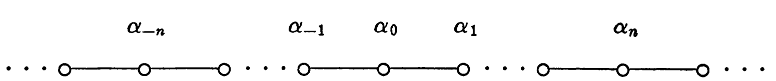
It is possible that all the β_j may be at one end of the chain. Again it follows that $\|h_{\alpha_i}\|=\|h_{\beta_j}\|$ for α_i, β_j. Continuing this process and introducing the necessary new notation for the α's in S we will obtain one of the following two possibilities:

Type A.



or

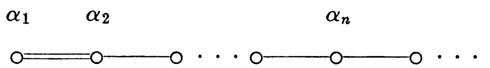
Type A'.



In either case $\|h_{\alpha_i}\| = \|h_{\alpha_j}\|$ for all i, j, $k_2(\alpha_i, \alpha_j) = 0$ for $j \neq i-1, i+1$ while $k_2(\alpha_i, \alpha_{i-1}) = k_2(\alpha_i, \alpha_{i+1}) = 1$. Thus the graph is completely determined up to a constant multiple.

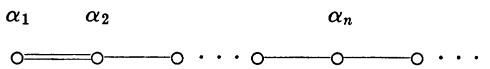
Entirely similar arguments for the other possibilities give the following types:

Type B.



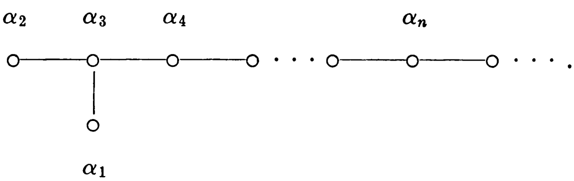
Here $2^{1/2}\|h_{\alpha_1}\| = \|h_{\alpha_i}\|$ for $i = 2, 3, \cdots$ and $k_2(\alpha_1, \alpha_2) = 1$, $k_2(\alpha_2, \alpha_1) = 1$ while otherwise $k_2(\alpha_i, \alpha_j)$ is as above.

Type C.



Here $\|h_{\alpha_1}\| = 2^{1/2}\|h_{\alpha_i}\|$ for $i = 2, 3, \cdots$ and $k_2(\alpha_1, \alpha_2) = 2$, $k_2(\alpha_2, \alpha_1) = 1$ while otherwise $k_2(\alpha_i, \alpha_j)$ is as above.

Type D.



Here $\|h_{\alpha_i}\| = \|h_{\alpha_j}\|$ for all i and j and $k_2(\alpha_1, \alpha_2) = k(\alpha_2, \alpha_1) = 0$, $k_2(\alpha_3, \alpha_1) = 1$ while otherwise $k_2(\alpha_i, \alpha_j)$ is as above.

3.6. In this paragraph it will be shown that each of the five types A, A′, B, C, D occurs as the graph of an L^* algebra. However, these algebras are not all distinct and give rise to only three nonisomorphic types. More explicitly, A and A′ are isomorphic and so are B and D.

All of these examples are Lie subalgebras of the associative H^* algebra K of Hilbert-Schmidt operators on a separable Hilbert space $\mathcal{H}$. For descriptive purposes it is convenient to choose an orthonormal basis of $\mathcal{H}$ and regard K as a matrix algebra relative to this basis. In each case a Cartan subalgebra H is obtained by taking the intersection of the algebra in question with the set of diagonal matrices. Having done this we will let λ_i denote the linear functional on H which assigns the ith diagonal entry to every element of H. Determination of a set of simple roots and the associated graph is analogous to the finite-dimensional case and the computations will be omitted here. After choosing the proper norm on L an application of 3.4 and 3.5 will show that L is isomorphic in all respects to one of the algebras described here.

In the following discussion a conjugate linear transformation J of $\mathcal{H}$ onto

$\mathcal{H}$ such that $(Jx, Jy)=(y, x)$ will be called a conjugation if $J^2=1$ and an anti-conjugation if $J^2=-1$.

TYPE A. Let $\{\phi_n: n=1, 2, \cdots\}$ be a basis of $\mathcal{H}$ and let A be the Lie algebra of all Hilbert-Schmidt matrices relative to this basis. A is simple since the center of K is trivial. A simple system of roots is given by

$$\{\lambda_i - \lambda_{i+1}: i = 1, 2, \cdots\}.$$

TYPE A′. Let $\{\phi_n: n=0, \pm 1, \pm 2, \cdots\}$ be a basis of $\mathcal{H}$ and let A′ be the Lie algebra of all Hilbert-Schmidt matrices relative to this basis. A simple system of roots is given by $\{\lambda_i-\lambda_{i+1}: i=0, \pm 1, \pm 2, \cdots\}$.

The algebras A and A' are isomorphic since there is a unitary operator U on $\mathcal{H}$ such that $X\in A$ if and only if UXU^{-1} is in A'.

TYPE B. Let $\{\phi_n: n=0, \pm 1, \pm 2, \cdots\}$ be a basis of $\mathcal{H}$ and let J_1 be the conjugation of $\mathcal{H}$ such that $J_1\phi_n=\phi_{-n}$. Let B be the set of Hilbert-Schmidt operators T such that $T^*J_1=-J_1T$. If $\langle x, y\rangle$ is defined by $\langle x, y\rangle=(x, J_1y)$ for $x, y\in\mathcal{H}$ then $\langle\ ,\ \rangle$ is a symmetric bilinear form and B is the set of T in K which are skew-adjoint with respect to this form. A simple system of roots is given by $\{\lambda_1, \lambda_i-\lambda_{i+1}: i=1, 2, \cdots\}$.

TYPE D. Let $\{\phi_n: n=\pm 1, \pm 2, \cdots\}$ be a basis of $\mathcal{H}$ and let J_2 be the conjugation on $\mathcal{H}$ such that $J_2\phi_n=\phi_{-n}$. Let D be the set of T in K such that $T^*J_2=-J_2T$. A simple system of roots is given by

$$\{\lambda_1 + \lambda_2, \lambda_i - \lambda_{i+1}: i = 1, 2, \cdots\}.$$

Since J_1 and J_2 are two conjugations of $\mathcal{H}$ there is a unitary U on $\mathcal{H}$ such that $UJ_1=J_2U$. Then for any $T\in K$, $T\in B$ if and only if UTU^{-1} is in D. Hence B is isomorphic to D.

TYPE C. Let $\{\phi_n: n=\pm 1, \pm 2, \cdots\}$ be a basis of $\mathcal{H}$. Let J be the anti-conjugation on $\mathcal{H}$ such that $J\phi_n=-\phi_{-n}$ for all positive n. Let C be the set of all Hilbert-Schmidt operators on $\mathcal{H}$ such that $T^*J=-JT$. Then C is the set of all $T\in K$ which are skew-symmetric with respect to the skew-symmetric form $\langle x, y\rangle=(x, Jy)$. A simple system of roots is given by

$$\{2\lambda_1, \lambda_i - \lambda_{i+1}: i = 1, 2, \cdots\}.$$

3.7. THEOREM 3. *Let L be a separable simple L^* algebra which has a Cartan decomposition relative to some Cartan subalgebra. Then (up to a multiple of the inner product on L) L is isomorphic to one of the following algebras:*

(1) *A, the algebra of all Hilbert-Schmidt operators on a separable Hilbert space $\mathcal{H}$.*

(2) *B, the algebra of all Hilbert-Schmidt operators T on $\mathcal{H}$ such that $T^*J=-JT$ for some fixed conjugation J of $\mathcal{H}$.*

(3) *C, the algebra of all Hilbert-Schmidt operators T on $\mathcal{H}$ such that $T^*J=-JT$ for some fixed anti-conjugation J of $\mathcal{H}$.*

REMARK. It still should be shown that the remaining three algebras A,

B, C are nonisomorphic. For two algebras L, L' of the type described in 3.7 and acting on the same space $\mathfrak{H}$ let L be equivalent to L' if and only if there is a unitary U on $\mathfrak{H}$ with $ULU^{-1}=L'$. We will show that L and L' are isomorphic only if they are equivalent. Since A, B, and C are clearly not equivalent this will be sufficient.

An $x \in L$ will be called primitive if (i) $x=x^* \neq 0$, (ii) $D_x^3 = D_x$, and (iii) x cannot be written $x=y+z$ where y and z satisfy (i) and (ii). By using the fact (see the proof of Theorem 2) that every Cartan subalgebra of L is a set of diagonal matrices relative to some basis of $\mathfrak{H}$ it follows that each such subalgebra has a basis of primitive elements and the vectors h_α are obtained from these by linear operations in a unique way according to the type of the associated graph. Since any isomorphism of L will preserve primitive elements the set $\{h_\alpha : \alpha \in R\}$, and hence the graph of L, is determined up to equivalence and the same will then hold for L.

4. Some remarks on derivations.

Definition. Let L be a semi-simple L^* algebra. A bounded operator D on L will be called a derivation of L if and only if $D[x, y] = [Dx, y] + [x, Dy]$ for all x, y in L.

If dim L is finite it is known that every derivation of L is inner, i.e. equal to D_x for some $x \in L$ [3, Exposé 7]. However, this is not true in general. To see this let A be the L^* algebra of all Hilbert-Schmidt operators on a separable infinite-dimensional Hilbert space. Then A is an associative ideal in the algebra of all bounded operators (see [4, pp. 73–75]). For a bounded operator B let T_B be the operator on A defined by $T_B X = BX - XB$. Then T_B is a bounded derivation of A and $T_B = 0$ if and only if B is a scalar multiple of the identity. Hence T_B is inner only if it differs from a Hilbert-Schmidt operator by a multiple of the identity and this implies A has outer derivations. Similar arguments can be used for B and C.

The same example can be used to show that the image of L under the adjoint representation need not be closed. By the closed graph theorem this is equivalent to proving that the norms on L and its image are not equivalent. Letting $L=A$ as above and regarding A as a matrix algebra with the usual unit matrices as a basis let $X_k = k^{-1/2} \sum_1^k E_{ii}$. Then $\|X_k\| = 1$ while $\|D_{X_k}\| = k^{-1/2}$. Thus $\|D_{X_k}\|$ tends to zero as k becomes large.

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